Bound of the density of the weighted non-centred χ^2 distribution

Let ξ be a Gaussian random element in $\mathbb{R}^p$ with zero mean and covariance matrix Σ. Denote by $\lambda_1 \geq \lambda_2 \geq \dots \lambda_p$ the eigenvalues of Σ. Let $F(x) := \mathbb{P}(\|\xi\|^2 \leq x)$ and define the density function $g(x) = F'(x)$ of $\|\xi\|^2$.

(1) Prove the following conjecture:

$$g(x) \leq c \prod_{j=1}^{2} \Psi_j^{-1/4},$$

where

$$\Psi_j := \sum_{k=j}^{p} \lambda_j^2,$$

and c is some absolute constant.

(2) Try to extend the result and consider the estimate of the derivative $g'(x)$.

Such bounds play a crucial role in the Gaussian comparison and anti-concentration inequalities which are important tools in many statistical applications, e.g. in non-parametric Bayes approach or bootstrap methods. See for example [2][Lemma 5.4], [1][Lemma A.1].

Prerequisites: basic knowledge of probability theory and analysis

References

[1] A. F. Götze, Naumov, V. Spokoiny, and V. Ulyanov. Gaussian comparison and anti-concentration inequlities for norms of Gaussian random vectors, 2017. arXiv:arXiv:1708.08663.

[2] A. Naumov, V. Spokoiny, and V. Ulyanov. Bootstrap confidence sets for spectral projectors of sample covariances, 2017. arXiv:1703.00871.