Central limit theorem in high-dimension

Let us consider a sample $X_1, \dots, X_n$ of independent identically distributed mean zero random vectors in high dimensional space $\mathbb{R}^p$. Denote by X_{ij} the j-th coordinate of X_i, so that $X_i = (X_{i1}, \dots, X_{ip})'$. Assume that $\mathbb{E}\, X_{ij}^2 = 1$ and $|X_{ij}| \le B$ for some constant B and $i = 1, \dots, n, j = 1, \dots, p$. Define the normalized sum

$$S_n := (S_{n,1}, \dots, S_{n,p})' := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_j.$$

We consider Gaussian approximation to S_n, and to this end, let $Y_1, \dots, Y_n$ be independent centered Gaussian random vectors in $\mathbb{R}^p$ such that each Y_i has the same covariance matrix as X_i, that is, $Y_i \sim \mathcal{N}(0, \mathbb{E}[X_i X_i'])$. Define the normalized sum for the Gaussian random vectors:

$$Z_n := (Z_{n,1}, \dots, Z_{n,p})' := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} Y_j.$$

It was recently proved in [2] that there exists absolute constant $c > 0$ such that

$$|\,\mathbb{P}(\max_{1\le j\le p} S_{n,j} \le x) - \mathbb{P}(\max_{1\le j\le p} Z_{n,j} \le x)| \le \left(\frac{cB^2 \log^a(pn)}{n}\right)^{1/6}, \tag{0.1}$$

where the current choice of a is $a = 7$. In particular this result implies that one may allow p to be of exponential order $\overline{o}(\exp(n^{1/7}))$.

(1) Prove (1.1) with $a = 5$.
(2) Try to extend to the case $a = 3$ (the conjectured optimal value of a).
(3) Study the moderate deviations, i.e. find the largest possible c_n so that

$$\lim_{n\to\infty} \frac{\mathbb{P}(\max_{1\le j\le p} S_{n,j} \ge x)}{\mathbb{P}(\max_{1\le j\le p} Z_{n,j} \ge x)} = 1$$

uniformly in $x \in [0, c_n]$. We refer to the recent monograph [1] for further reading.

Prerequisites: advanced knowledge of probability theory, analysis, linear algebra

References

[1] L. Chen, L. Goldstein, and Q.M. Shao. *Normal approximation by Stein's method.* Probability and its Applications (New York). Springer, Heidelberg, 2011.

[2] V. Chernozhukov, D. Chetverikov, and K. Kato. Central limit theorems and bootstrap in high dimensions. *Ann. Probab.*, 45(4):2309–2352, 2017.